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Research Article

Almost Sure Central Limit Theorem for Product of Partial Sums of Strongly Mixing Random Variables

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We give here an almost sure central limit theorem for product of sums of strongly mixing positive random variables.

1. Introduction and Results

In recent decades, there has been a lot of work on the almost sure central limit theorem (ASCLT), we can refer to Brosamler [1], Schatte [2], Lacey and Philipp [3], and Peligrad and Shao [4].

Khurelbaatar and Rempala [5] gave an ASCLT for product of partial sums of i.i.d. random variables as follows.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive random variables with $EX_1 = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x*

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} I\left(\left(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k}\right)^{1/\gamma\sqrt{k}} \leq x\right) = F(x) \quad a.s., \tag{1.1}$$

where $S_n = \sum_{k=1}^{n} X_k$, $I(*)$ is the indicator function, $F(\cdot)$ is the distribution function of the random variable $e^{\mathcal{N}}$, and $\mathcal{N}$ is a standard normal variable.

Recently, Jin [6] had proved that (1.1) holds under appropriate conditions for strongly mixing positive random variables and gave an ASCLT for product of partial sums of strongly mixing as follows.

Theorem 1.2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed positive strongly mixing random variable with $EX_1 = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$, $d_k = 1/k$, $D_n = \sum_{k=1}^{n} d_k$. Denote by $\gamma = \sigma/\mu$ the coefficient of variation, $\sigma_n^2 = \operatorname{Var}(\sum_{k=1}^{n}((S_k - k\mu)/k\sigma))$ and $B_n^2 = \operatorname{Var}(S_n)$. Assume*

$$\begin{aligned} &E|X_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \qquad \lim_{n\to\infty} \frac{B_n^2}{n} = \sigma_0^2 > 0, \\ &\alpha(n) = O\left(n^{-r}\right) \quad \text{for some } r > 1 + \frac{2}{\delta}, \qquad \inf_{n\in N} \frac{\sigma_n^2}{n} > 0. \end{aligned} \tag{1.2}$$

Then for any real x

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left(\left(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k}\right)^{1/\gamma\sigma_k} \leq x\right) = F(x) \quad a.s. \tag{1.3}$$

The sequence $\{d_k, k \geq 1\}$ in (1.3) is called weight. Under the conditions of Theorem 1.2, it is easy to see that (1.3) holds for every sequence d_k^* with $0 \leq d_k^* \leq d_k$ and $D_n^* = \sum_{k\leq n} d_k^* \to \infty$ [7]. Clearly, the larger the weight sequence (d_k) is, the stronger is the result (1.3).

In the following sections, let $d_k = e^{(\ln k)^\alpha}/k$, $0 \leq \alpha < 1/2$, $D_n = \sum_{k=1}^{n} d_k$, "$\ll$" denote the inequality "$\leq$" up to some universal constant.

We first give an ASCLT for strongly mixing positive random variables.

Theorem 1.3. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed positive strongly mixing random variable with $EX_1 = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$, d_k and D_n as mentioned above. Denote by $\gamma = \sigma/\mu$ the coefficient of variation, $\sigma_n^2 = \operatorname{Var}(\sum_{k=1}^{n}((S_k - k\mu)/k\sigma))$ and $B_n^2 = \operatorname{Var}(S_n)$. Assume that*

$$E|X_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \tag{1.4}$$

$$\alpha(n) = O\left(n^{-r}\right) \quad \text{for some } r > 1 + \frac{2}{\delta}, \tag{1.5}$$

$$\lim_{n\to\infty} \frac{B_n^2}{n} = \sigma_0^2 > 0, \tag{1.6}$$

$$\inf_{n\in N} \frac{\sigma_n^2}{n} > 0. \tag{1.7}$$

Then for any real x

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left(\left(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k}\right)^{1/\gamma\sigma_k} \leq x\right) = F(x) \quad a.s. \tag{1.8}$$

In order to prove Theorem 1.3 we first establish ASCLT for certain triangular arrays of random variables. In the sequel we shall use the following notation. Let $b_{k,n} = \sum_{i=k}^{n}(1/i)$ and $s_{k,n}^2 = \sum_{i=1}^{k} b_{i,n}^2$ for $k \leq n$ with $b_{k,n} = 0$ if $k > n$. $Y_k = (X_k - \mu)/\sigma$, $k \leq 1$, $\widetilde{S}_n = \sum_{k=1}^{n} Y_k$ and $S_{n,n} = \sum_{k=1}^{n} b_{k,n} Y_k$.

In this setting we establish an ASCLT for the triangular array $(b_{k,n}Y_k)$.

Theorem 1.4. *Under the conditions of Theorem 1.3, for any real x*

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left\{\frac{S_{k,k}}{\sigma_k}\le x\right\}=\Phi(x)\quad a.s.,\tag{1.9}$$

where $\Phi(x)$ is the standard normal distribution function.

2. The Proofs

2.1. Lemmas

To prove theorems, we need the following lemmas.

Lemma 2.1 (see [8]). *Let $\{X_n, n\ge 1\}$ be a sequence of strongly mixing random variables with zero mean, and let $\{a_{k,n}, 1\le k\le n, n\ge 1\}$ be a triangular array of real numbers. Assume that*

$$\sup_n\sum_{k=1}^{n}a_{k,n}^2<\infty,\qquad \max_{1\le k\le n}|a_{k,n}|\longrightarrow 0\quad as\ n\longrightarrow\infty.\tag{2.1}$$

If for a certain $\delta>0$, $\{|X_k|^{2+\delta}\}$ is uniformly integrable, $\inf_k \operatorname{Var}(X_k)>0$,

$$\sum_{n=1}^{\infty}n^{2/\delta}\alpha(n)<\infty,\qquad \operatorname{Var}\left(\sum_{n=1}^{n}a_{k,n}X_k\right)=1,\tag{2.2}$$

then

$$\sum_{k=1}^{n}a_{k,n}X_k\xrightarrow{d}\mathcal{N}(0,1).\tag{2.3}$$

Lemma 2.2 (see [9]). *Let $d_k=\mathrm{e}^{(\ln k)^\alpha}/k,\ 0\le\alpha<1/2, D_n=\sum_{k=1}^{n}d_k$; then*

$$D_n\sim C(\ln n)^{1-\alpha}\exp\{(\ln n)^\alpha\},\tag{2.4}$$

where $C=1/\alpha$ as $0<\alpha<1/2, C=1$ as $\alpha=0$.

Lemma 2.3 (see [8]). *Let $\{X_n, n\ge 1\}$ be a strongly mixing sequence of random variables such that $\sup_n E|X_n|^{2+\delta}<\infty$ for a certain $\delta>0$ and every $n\ge 1$. Then there is a numerical constant $c(\delta)$ depending only on δ such that for every $n>1$ one has*

$$\sup_j\sum_{i=j+1}^{n+j}\left|\operatorname{Cov}(X_i,X_j)\right|\le c(\delta)\left(\sum_{i=1}^{n}i^{2/\delta}\alpha(i)\right)^{\delta/(2+\delta)}\sup_k\|X_k\|_{2+\delta}^2,\tag{2.5}$$

where $\|X_k\|_p=E(|X_k|^p)^{1/p},\ p>1$.

Lemma 2.4 (see [9]). *Let $\{\xi_k, k \geq 1\}$ be a sequence of random variables, uniformly bounded below and with finite variances, and let $\{d_k, k \geq 1\}$ be a sequence of positive number. Let for $n \geq 1$, $D_n = \sum_{k=1}^{n} d_k$ and $T_n = (1/D_n)\sum_{k=1}^{n} d_k \xi_k$. Assume that*

$$D_n \longrightarrow \infty \quad \frac{D_{n+1}}{D_n} \longrightarrow 1, \tag{2.6}$$

as $n \to \infty$. If for some $\varepsilon > 0$, C and all n

$$ET_n^2 \leq C\left(\ln^{-1-\varepsilon} D_n\right), \tag{2.7}$$

then

$$T_n \xrightarrow{a.s.} 0 \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

Lemma 2.5 (see [10]). *Let $\{X_n, n \geq 1\}$ be a strongly mixing sequence of random variables with zero mean and $\sup_n E|X_n|^{2+\delta} < \infty$ for a certain $\delta > 0$. Assume that* (1.5) *and* (1.6) *hold. Then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2\sigma_0^2 n \ln\ln n}} = 1 \quad \text{a.s.} \tag{2.9}$$

2.2. Proof of Theorem 1.4

From the definition of strongly mixing we know that $\{Y_k, k \geq 1\}$ remain to be a sequence of identically distributed strongly mixing random variable with zero mean and unit variance. Let $a_{k,n} = b_{k,n}/\sigma_n$; note that

$$\sum_{k=1}^{n} b_{k,n}^2 = b_{1,n} + 2\sum_{k=2}^{n}\sum_{i=1}^{k-1}\frac{1}{k} = b_{1,n} + 2\sum_{k=2}^{n}\frac{k-1}{k} = 2n - b_{1,n}, \quad n \geq 1, \tag{2.10}$$

and via (1.7) we have

$$\begin{aligned}
&\sup_n \sum_{k=1}^{n} a_{k,n}^2 = \sup_n \sum_{k=1}^{n} \frac{b_{k,n}^2}{\sigma_n^2} \ll \sup_n \frac{2n - b_{1,n}}{n} < \infty, \\
&\max_{1\leq k\leq n} |a_{k,n}| = \max_{1\leq k\leq n} \frac{b_{k,n}}{\sigma_n} \ll \frac{\ln n}{\sqrt{n}} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{2.11}$$

From the definition of Y_k and (1.4) we have that $\{|Y_k|^{2+\delta}\}$ is uniformly integrable; note that

$$\inf_k \operatorname{Var}(Y_k) = EY_1^2 = 1 > 0, \qquad \operatorname{Var}\left(\sum_{k=1}^{n} a_{k,n} Y_k\right) = \frac{\operatorname{Var}\left(\sum_{k=1}^{n} b_{k,n} Y_k\right)}{\sigma_n^2} = 1, \tag{2.12}$$

and applying (1.5)

$$\sum_{n=1}^{\infty} n^{2/\delta}\alpha(n) \ll \sum_{n=1}^{\infty} n^{-r+2/\delta} < \infty. \tag{2.13}$$

Consequently using Lemma 2.1, we can obtain

$$\frac{S_{n,n}}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \longrightarrow \infty, \tag{2.14}$$

which is equivalent to

$$Ef\left(\frac{S_{n,n}}{\sigma_n}\right) \longrightarrow Ef(\mathcal{N}) \quad \text{as } n \longrightarrow \infty \tag{2.15}$$

for any bounded Lipschitz-continuous function f; applying *Toeplitz Lemma*

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k Ef\left(\frac{S_{k,k}}{\sigma_k}\right) \longrightarrow Ef(\mathcal{N}) \quad \text{as } n \longrightarrow \infty. \tag{2.16}$$

We notice that (1.9) is equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^{n} d_k f\left(\frac{S_{k,k}}{\sigma_k}\right) = \Phi(x) \quad \text{a.s.} \tag{2.17}$$

for all bounded Lipschitz continuous f; it therefore remains to prove that

$$T_n \triangleq \frac{1}{D_n}\sum_{k=1}^{n} d_k\left(f\left(\frac{S_{k,k}}{\sigma_k}\right) - Ef\left(\frac{S_{k,k}}{\sigma_k}\right)\right) \xrightarrow{\text{a.s.}} 0, \quad n \longrightarrow \infty. \tag{2.18}$$

Let $\xi_k = f(S_{k,k}/\sigma_k) - Ef(S_{k,k}/\sigma_k)$,

$$\begin{aligned} E\left(\sum_{k=1}^{n} d_k\xi_k\right)^2 &\le E\left(2\sum_{1\le k\le l\le n} d_k d_l \xi_k \xi_l\right) \ll \sum_{1\le k\le l\le n} d_k d_l |E(\xi_k\xi_l)| \\ &= \sum_{\substack{1\le k\le l\le n \\ l\le 2k}} d_k d_l |E(\xi_k\xi_l)| + \sum_{\substack{1\le k\le l\le n \\ l>2k}} d_k d_l |E(\xi_k\xi_l)| \\ &\triangleq T_{1,n} + T_{2,n}. \end{aligned} \tag{2.19}$$

From Lemma 2.2, we obtain for some constant C_1

$$\mathrm{e}^{(\ln n)^{\alpha}} \sim C_1 D_n (\ln D_n)^{1-1/\alpha}. \tag{2.20}$$

Using (2.20) and property of f, we have

$$T_{1,n} \ll \mathrm{e}^{(\ln n)^{\alpha}} \sum_{k=1}^{n} d_k \sum_{l=k}^{2k} \frac{1}{l} \ll D_n \mathrm{e}^{(\ln n)^{\alpha}} \ll D_n^2 (\ln D_n)^{1-1/\alpha}. \tag{2.21}$$

We estimate now $T_{2,n}$. For $l > 2k$,

$$\begin{aligned} S_{l,l} - S_{2k,2k} &= (b_{1,l}Y_1 + b_{2,l}Y_2 + \cdots + b_{l,l}Y_l) - (b_{1,2k}Y_1 + b_{2,2k}Y_2 + \cdots + b_{2k,2k}Y_{2k}) \\ &= b_{2k+1,l}\widetilde{S}_{2k} + (b_{2k+1,l}Y_{2k+1} + \cdots + b_{l,l}Y_l). \end{aligned} \tag{2.22}$$

Notice that

$$\begin{aligned} |E\xi_k\xi_l| &= \left|\operatorname{Cov}\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right)\right)\right| \\ &\le \left|\operatorname{Cov}\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right) - f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\widetilde{S}_{2k}}{\sigma_l}\right)\right)\right| \\ &\quad + \left|\operatorname{Cov}\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\widetilde{S}_{2k}}{\sigma_l}\right)\right)\right|, \end{aligned} \tag{2.23}$$

and the properties of strongly mixing sequence imply

$$\left|\operatorname{Cov}\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\widetilde{S}_{2k}}{\sigma_l}\right)\right)\right| \ll \alpha(k). \tag{2.24}$$

Applying Lemma 2.3 and (2.10),

$$\begin{aligned} \operatorname{Var}(S_{2k,2k}) &= \sum_{i=1}^{2k} b_{i,2k}^2 EY_i^2 + 2\sum_{j=1}^{2k-1} \sum_{i=j+1}^{2k} b_{i,2k} b_{j,2k} \operatorname{Cov}(Y_i, Y_j) \\ &\le \sum_{i=1}^{2k} b_{i,2k}^2 + 2\sum_{j=1}^{2k-1} b_{j,2k}^2 \sum_{i=j+1}^{2k} \left|\operatorname{Cov}(Y_i, Y_j)\right| \ll k, \\ \operatorname{Var}\left(\widetilde{S}_{2k}\right) &= E\left(\sum_{i=1}^{2k} Y_i\right)^2 = \sum_{i=1}^{2k} EY_i^2 + 2\sum_{i=1}^{2k-1} \sum_{j=i+1}^{2k} \operatorname{Cov}(Y_i, Y_j) \ll k. \end{aligned} \tag{2.25}$$

Consequently, via the properties of f, the *Jensen* inequality, and (1.7),

$$\begin{aligned} &\left|\operatorname{Cov}\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right) - f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right| \\ &\quad \ll \frac{E\left|S_{2k,2k} + b_{2k+1,l}\tilde{S}_{2k}\right|}{\sigma_l} \le \frac{\sqrt{ES_{2k,2k}^2}}{\sigma_l} + \frac{\sqrt{E\left(b_{2k+1,l}\tilde{S}_{2k}\right)^2}}{\sigma_l} \\ &\quad = \frac{\sqrt{\operatorname{Var}(S_{2k,2k})}}{\sigma_l} + b_{2k+1,l}\frac{\sqrt{\operatorname{Var}\left(\tilde{S}_{2k}\right)}}{\sigma_l} \ll \left(\frac{k}{l}\right)^{\beta}, \end{aligned} \tag{2.26}$$

where $0 < \beta < 1/2$. Hence for $l > 2k$ we have

$$|E\xi_k\xi_l| \ll \alpha(k) + \left(\frac{k}{l}\right)^{\beta}. \tag{2.27}$$

Consequently, we conclude from the above inequalities that

$$\begin{aligned} T_{2,n} &\ll \sum_{\substack{1\le k\le l\le n \\ l>2k}} d_k d_l \left(\alpha(k) + \left(\frac{k}{l}\right)^{\beta}\right) \\ &= \sum_{\substack{1\le k\le l\le n \\ l>2k}} d_k d_l \alpha(k) + \sum_{\substack{1\le k\le l\le n \\ l>2k}} d_k d_l \left(\frac{k}{l}\right)^{\beta} \triangleq T_{2,n,1} + T_{2,n,2}. \end{aligned} \tag{2.28}$$

Applying (1.5) and Lemma 2.2 we can obtain for any $\eta > 0$

$$T_{2,n,1} \le \sum_{k=1}^{n}\sum_{l=1}^{n} d_k d_l \alpha(k) \ll (\ln D_n)^{-1-\eta}\sum_{k=1}^{n} d_k \sum_{l=1}^{n} d_l = D_n^2 (\ln D_n)^{-1-\eta}. \tag{2.29}$$

Notice that

$$T_{2,n,2} = \sum_{\substack{1\le k\le l\le n \\ l>2k \\ (l/k)\ge(\ln D_n)^{2/\beta}}} d_k d_l \left(\frac{k}{l}\right)^{\beta} + \sum_{\substack{1\le k\le l\le n \\ l>2k \\ (l/k)<(\ln D_n)^{2/\beta}}} d_k d_l \left(\frac{k}{l}\right)^{\beta} \triangleq T_{2,n,2,1} + T_{2,n,2,2}, \tag{2.30}$$

$$T_{2,n,2,1} \le \sum_{\substack{1\le k\le l\le n \\ l>2k}} d_k d_l (\ln D_n)^{-2} \le (\ln D_n)^{-2}\sum_{k=1}^{n} d_k \sum_{l=1}^{n} d_l = D_n^2 (\ln D_n)^{-2}. \tag{2.31}$$

Let $n_0 = \max\{l : k \le l \le n,\ (l/k) < (\ln D_n)^{2/\beta}\}$, then

$$\begin{aligned} T_{2,n,2,2} &\le \sum_{k=1}^{n}\sum_{l=2k}^{n_0} d_k d_l \le e^{(\ln n)^\alpha}\sum_{k=1}^{n} d_k \sum_{l=2k}^{n_0}\frac{1}{l} \ll e^{(\ln n)^\alpha}\sum_{k=1}^{n} d_k(\ln n_0 - \ln 2k) \\ &\ll e^{(\ln n)^\alpha} D_n \ln\ln D_n \ll D_n^2 \ln^{1-1/\alpha} D_n \ln\ln D_n. \end{aligned} \tag{2.32}$$

By (2.21), (2.29), (2.31), and (2.32), for some $\varepsilon > 0$ such that

$$ET_n^2 = \frac{1}{D_n^2} E\left(\sum_{k=1}^{n} d_k \xi_k\right)^2 \ll (\ln D_n)^{-1-\varepsilon}, \tag{2.33}$$

applying Lemma 2.4, we have

$$T_n \xrightarrow{\text{a.s.}} 0. \tag{2.34}$$

2.3. *Proof of Theorem 1.3*

Let $C_k = S_k/\mu k$; we have

$$\frac{1}{\gamma\sigma_n}\sum_{k=1}^{n}(C_k - 1) = \frac{1}{\gamma\sigma_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k} - 1\right) = \frac{1}{\sigma_n}\sum_{k=1}^{n} b_{k,n} Y_k = \frac{S_{n,n}}{\sigma_n}. \tag{2.35}$$

We see that (1.9) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{1}{\gamma\sigma_k}\sum_{i=1}^{k}(C_i - 1) \le x\right) = \Phi(x), \quad \text{a.s. } \forall x. \tag{2.36}$$

Note that in order to prove (1.8) it is sufficient to show that

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{1}{\gamma\sigma_k}\sum_{i=1}^{k}\ln C_i \le x\right) = \Phi(x), \quad \text{a.s. } \forall x. \tag{2.37}$$

From Lemma 2.5, for sufficiently large k, we have

$$|C_k - 1| = O\left(\left(\frac{\ln(\ln k)}{k}\right)^{1/2}\right). \tag{2.38}$$

Since $\ln(1+x) = x + O(x^2)$ for $|x| < 1/2$, thus

$$\left|\sum_{k=1}^{n}\ln(C_k) - \sum_{k=1}^{n}(C_k - 1)\right| \ll \sum_{k=1}^{n}(C_k - 1)^2 \ll \sum_{k=1}^{n}\frac{\ln(\ln k)}{k} \ll \ln n \ln(\ln n) \quad \text{a.s.} \tag{2.39}$$

Hence for any $\varepsilon > 0$ and for sufficiently large n, we have

$$I\left(\frac{1}{\gamma\sigma_n}\sum_{k=1}^{n}(C_k - 1) \leq x - \varepsilon\right) \leq I\left(\frac{1}{\gamma\sigma_n}\sum_{k=1}^{n}\ln C_k \leq x\right) \leq I\left(\frac{1}{\gamma\sigma_n}\sum_{k=1}^{n}(C_k - 1) \leq x + \varepsilon\right) \tag{2.40}$$

and thus (2.36) implies (2.37).

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